Estimation Enhancement by Imposing a Relative Intercept Angle for Defending Missiles

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A multiple-body interception scenario is considered, where an evading target aircraft launches two defending missiles as a countermeasure against an incoming homing missile. The defenders share their respective line of sight angle measurements, are aware of the target's evasion strategy, and exploit a cooperate guidance law which can impose a relative intercept angle between them. The ability of the proposed cooperative guidance-estimation scheme to protect the targeted aircraft is analyzed. Especially, the effect of different values of the commanded relative intercept angle on the intertwined guidance-estimation performance is studied.

I. Introduction

A scenario where an attacking missile, homing onto an evading target aircraft, encounters two defending missiles launched by the aircraft is considered. The incoming missile employs a linear guidance law against the evading target. From the target's perspective, possible solutions to deal with such a challenge are to develop advanced sensors to accurately track the missile, to use more agile defenders, and/or to install a more lethal warhead for the defenders. These options might be very often too complex, heavy, and expensive. An alternative is to design more sophisticated guidance-estimation algorithms to improve the guidance system of inexpensive defenders so that they will be able to achieve the required interception accuracy without changing existing hardware. Therefore, we assume that the defenders are equipped with sensors which allow to measure only the line of sight (LOS) angle and that they have limited maneuver capabilities.

In scenarios where multiple vehicles can share their respective LOS angle measurements, the estimation performance can be improved by exploiting the triangulation method [1–3]. The estimation quality, however, strongly depends upon the vehicles' trajectories and hence on the implemented guidance strategy. In [1], two distinct estimator design methods for cooperative target tracking are presented. It was assumed that the vehicles (missiles) are guided to the target via non-cooperative guidance laws and that only the estimation is performed cooperatively. As guidance and estimation are mutually intertwined, neglecting the effect of the one onto the other and vice versa may have severe consequences. For example, if all vehicles employ the same one-on-one guidance law (as considered in [1]) and are all launched with the same initial conditions, then the resulting trajectories coincide (up to some unmodeled disturbances). As a consequence, all sensors will measure the same quantity (LOS angle), causing the triangulation technique to fail. This, in turn, might result in poor interception performance.

The work [1] was recently extended in [3] by the same authors, where the above issue was tackled by introducing the concept of staggered launch of the missiles. The observability issue in a double-LOS relative navigation setup was analyzed in [4, 5]. It was concluded that if the separation angle of the LOS vectors is too small, the relative navigation system may become weakly observable or even unobservable. This problem was addressed for the two missiles case in [2] by modulating the LOS angle through a performance index. The missile with large initial LOS angle maximizes this index while the other one minimizes it. By this, the separation angle of both LOS vectors during the engagement is increased and the estimation is improved.

In this paper, an intuitive approach for improving the estimation performance and thus the defending capabilities of the two defending missiles is to impose a nonzero relative intercept angle constraints for the

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defenders. This can be easily achieved for each defender separately using one of the available one-on-one terminal intercept angle guidance laws, see e.g., [6, 7], or, better yet and as done in this paper, one can achieve the same goal using the optimal cooperative guidance law presented in [8] which can impose a predetermined relative intercept angle between consecutive vehicles.

As larger intercept angles require more maneuverability and the fact that control saturation goes hand in hand with degradation in the guidance performance, it is important therefore to study the effect of estimation on to the guidance problem. In this paper, we rigorously analyze the effect of different values of the commanded relative intercept angle on the intertwined guidance-estimation performance.

The remainder of this paper is organized as follows. The next section presents the mathematical models of the target-defenders-missile engagement. The guidance laws are presented in Sec. III, followed by the derivation of the estimator in Sec. IV. A comprehensive performance analysis is done in Sec. V, followed by concluding remarks.

II. Multiple-Body Engagement Description

The problem consists of four entities (also referred to as bodies or vehicles): an attacking missile, an evading target aircraft, and two defending missiles. For brevity, the target aircraft is referred as target, the attacking missile as missile, and the defending missiles as defenders. The defenders are launched simultaneously by the evading target to intercept the incoming threat. The attacking missile is unaware of the defenders and employs a known linear one-on-one guidance strategy to intercept the evading target.

II.A. Nonlinear Kinematics and Dynamics

We consider a skid-to-turn and roll-stabilized vehicles. The motion of the four bodies is assumed to transpire in the same plane. In Figure 1 a schematic view of the planar point mass target-defenders-missile engagement geometry is shown, where X_I - O_I - Y_I represents a Cartesian inertial reference frame. The missile, defender, and target related variables are denoted by the subscripts m, d, and t, respectively. The speed, normal acceleration, and flight-path angle are denoted by V, a, and γ, respectively. The range between the target and missile and between the i-th defender and missile is denoted as ρ_{tm} and $\rho_{d_i m}$, respectively. The angle between the target's line-of-sight (LOS) to the missile and the X_I axis is denoted as λ_{tm}, while that between the i-th defender's LOS to the missile and the X_I axis is denoted as $\lambda_{d_i m}$.

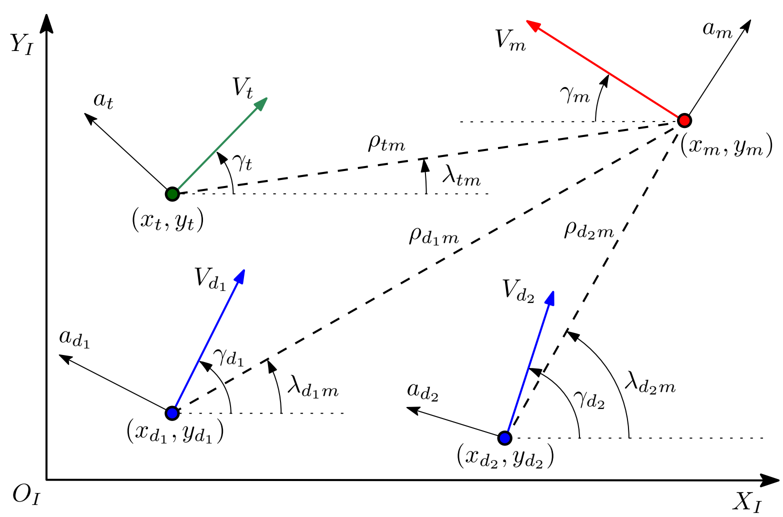


Figure 1: Planar target-defenders-missile engagement geometry.

The missile and the defenders are considered to be from a similar class of vehicles, with their speeds higher than of the target, that is, $V_i > V_t$, $\forall i \in \{d_1, d_2, m\}$. We assume that the target's and defenders' own inertial state vector

$$x_i^I = \begin{bmatrix} x_i & y_i & a_i & \gamma_i \end{bmatrix}^T, \qquad i \in \{t, d_1, d_2\} \tag{1}$$

is known to a very high accuracy (e.g., using inertial navigation system and/or GPS sensors), and that the target and each defender can transmit its own state vector to both defenders and to the other defender without any delay, respectively. The target's speed is assumed to be known is transmitted to the defenders.

Neglecting the gravity, the engagement kinematics, expressed in a polar coordinate system $(\rho_{im}, \lambda_{im})$, is:

$$\begin{cases} \dot{\rho}_{im} = V_{\rho im}, \\ \dot{\lambda}_{im} = V_{\lambda im}/\rho_{im}, \end{cases} \qquad i \in \{t, d_1, d_2\}, \tag{2}$$

where the respective relative velocities along and perpendicular to the LOS are

$$V_{\rho im} = -V_i \cos(\gamma_i - \lambda_{im}) - V_m \cos(\gamma_m + \lambda_{im}), \tag{3a}$$

$$V_{\lambda im} = -V_i \sin(\gamma_i - \lambda_{im}) + V_m \sin(\gamma_m + \lambda_{im}). \tag{3b}$$

During the endgame, all four vehicles are assumed to move at a constant speed and to perform lateral maneuvers only. Arbitrary-order linear dynamics is assumed for all four vehicles

$$\begin{cases} \dot{x}_i^a = A_i x_i^a + B_i u_i \\ a_i = C_i x_i^a + D_i u_i \ , \\ \dot{\gamma}_i = a_i / V_i \end{cases} \qquad i = \{t, d_1, d_2, m\}, \tag{4}$$

where $x_i^a \in \mathbb{R}^{n_i}$ is the internal state vector of the i-th vehicle's dynamics, a_i and u_i are the i-th entity's acceleration and acceleration command, respectively. The term $C_i x_i^a$ is denoted as a_{is} and represents, if it exists, the part of the acceleration with dynamics.

We assume that the target's and defenders' maneuverabilities are limited to

$$|u_i| \leq a_i^{max}, \qquad i \in \{t, d_1, d_2\}, \tag{5}$$

where $a_i^{max} > 0$ is the maximal acceleration of the i-th vehicle. No saturation is considered for the missile, thus $u_m^{max} = \infty$.

II.B. Timeline and Time-to-go

The running time is denoted as t. The endgame initiates at $t = 0$ with $\dot{\rho}_{im}(t = 0) < 0$, $\forall i \in \{t, d_1, d_2\}$ and the particular engagement terminates at $t = t_{im}^f$, where t_{im}^f is the target-missile or defender-missile interception time, formally defined as

$$t_{im}^f = \arg\inf_{t>0} \{\rho_{im}(t) V_{\rho im}(t) = 0\}, \qquad i \in \{t, d_1, d_2\}, \tag{6}$$

and allowing to define the nonnegative time-to-go by

$$t_{im}^{go} = \begin{cases} t_{im}^f - t, & t \leq t_{im}^f \\ 0, & t > t_{im}^f \end{cases} , \qquad i \in \{t, d_1, d_2\}. \tag{7}$$

At $t = t_{im}^f$, the separation $\rho_{im}(t_{im}^f)$ is minimal and is referred to as "miss distance" or compactly as "miss".

We require that the target-missile engagement terminates after that of the defenders-missile, therefore

$$t_{im}^f < t_{tm}^f, \quad i \in \{d_1, d_2\}. \tag{8}$$

II.C. Physical Measurement Model

As the bearing measurement is a predominant one in missile guidance applications, it is assumed that each defender is only equipped with an electro-optical seeker that measures the LOS angle λ_{im}, $i \in \{d_1, d_2\}$. Both measurements are assumed to be acquired at the same discrete-time $t = t_k \triangleq k \cdot T$, where T is the measurement sampling period. The measurements are corrupted by a zero-mean white Gaussian noise with standard deviation $\sigma_{\lambda i}$, $i \in \{d_1, d_2\}$.

The physical measurement equation of the i-th defender is

$$z_{i;k} = h_i(x_k) + v_{i;k} = \lambda_{im;k} + v_{i;k}, \qquad i \in \{d_1, d_2\}, \tag{9}$$

where the state vector x_k, used for estimation, will be defined later, and

$$v_{i;k} \sim \mathcal{N}(0, \sigma_{\lambda i}^2), \qquad i \in \{d_1, d_2\}.$$

In Eq. (9) and in the rest of the paper, the discrete time step is indicated by a subscript k, separated by a semicolon. The measurement noise sequences $v_{i;k}$, $i \in \{d_1, d_2\}$ are assumed to be mutually independent. It is assumed that the first defender can transmit its measurements to the second defender without any delay and vice versa. The advantage and utilization of such measurements sharing will be discussed in Sec. IV.B.

III. Guidance Laws

III.A. Perfect Information Guidance Law of the Missile

For simplicity, we will limit our derivation to three representative missile guidance laws of PN [9], APN [10], and OGL [11]. The derivation can be extended to other missile guidance laws using the same formulation and similar derivation steps.

The guidance law of PN, APN, and OGL is widely known in the following form

$$u_m = N_j' \frac{Z_j}{(t_{tm}^{go})^2 \cos(\gamma_m + \lambda_{tm})}, \qquad j \in \{\text{PN}, \text{APN}, \text{OGL}\}, \tag{10}$$

where N' is the effective navigation gain, Z is the missile's zero-effort-miss (ZEM) distance, and t_{tm}^{go} is given by

$$t_{tm}^{go} = -\rho_{tm}/V_{\rho tm}, \qquad V_{\rho tm} < 0. \tag{11}$$

The expression for the ZEM distance is different for each guidance law, as it is dependent on the model used and assumptions made regarding the future target maneuvers.

Under the assumptions of ideal missile dynamics and no target maneuver (i.e., $u_t = 0$), the obtained missile guidance law guaranteeing zero miss distance is PN with

$$Z_{PN} = V_{\rho tm} \dot{\lambda}_{tm} (t_{tm}^{go})^2, \tag{12}$$

where $\dot{\lambda}_{tm}$ and $V_{\rho tm}$ are given by Eqs. (2) and (3a), respectively. If $N_{PN}' = 3$, this guidance law minimizes the missile's control effort. Extending the results to the case in which the target is assumed to perform a constant maneuver (i.e., $u_t = \text{const.}$), APN was obtained with

$$Z_{APN} = Z_{PN} + \frac{(t_{tm}^{go})^2}{2} a_t \cos(\gamma_t - \lambda_{tm}). \tag{13}$$

Additionally assuming that the missile's closed-loop acceleration dynamics can be approximated by a first-order strictly proper transfer function with a time constant τ_m, OGL was obtained with

$$Z_{OGL} = Z_{APN} - \tau_m^2 \psi(\theta_{tm}) a_{ms} \cos(\gamma_m + \lambda_{tm}), \tag{14}$$

where $\psi(\theta_{tm})$ is an exponential-like function of the normalized target-missile time-to-go θ_{tm} given by

$$\psi(\theta_{tm}) = \exp(-\theta_{tm}) + \theta_{tm} - 1, \qquad \theta_{tm} = t_{tm}^{go}/\tau_m. \tag{15}$$

The navigation gains of PN and APN are constant, whereas that of OGL is time-varying and given by

$$N_{OGL}'(\theta_{tm}) = \frac{6\theta_{tm}^2 \psi(\theta_{tm})}{3 + 6\theta_{tm} - 6\theta_{tm}^2 + 2\theta_{tm}^3 - 3e^{-2\theta_{tm}} - 12\theta_{tm} e^{-\theta_{tm}} + 6\alpha/\tau_m^3}, \tag{16}$$

where α represents the ratio between the weights on the control effort and the miss distance in the quadratic cost function used in the OGL formulation.

III.B. Evasive Strategy of the Target

The optimal one-on-one perfect information evasion strategy against a homing missile employing a linear guidance law has a bang-bang structure for bounded target acceleration [12]. The practical implementation of such evasion strategy requires the estimate of the relative target-missile state and perfect knowledge about the active missile guidance law and guidance parameters [13].

Note that the optimal one-on-one evasion strategy is not necessary the "best" from the defenders point of view [14, 15]. The target's guidance directly influences the homing missile's trajectory, and hence, indirectly, also the trajectories of the defenders. Thus, the target may lure the missile such that the defenders can intercept it even when the defenders' maneuver capability is much smaller compared to that of the missile or the target can help to improve the estimation performance of the defenders. In this paper, we assume that the evasion strategy of the target is arbitrary and that it is perfectly known to both defenders.

III.C. Cooperative Guidance Law of the Defenders

Here we present the guidance strategy for the defender team. The cornerstone of this strategy is based in the recently developed cooperative optimal guidance law for imposing a relative intercept angle between consecutive vehicles, see [8]. This law ensures cooperation between the two defenders (explicit cooperation) and the protected target aircraft (implicit cooperation).

Explicit Cooperation to Impose a Relative Intercept Angle

To help the estimation process, we wish to impose a relative intercept angle between the defenders. Denote the angle between the i-th defender and the missile as $\gamma_{d_i m} = \gamma_{d_i} + \gamma_m$. The difference between the intercept angles $\gamma_{d_1 m}$ and $\gamma_{d_2 m}$ is the relative intercept angle from the missile's perspective. This is the angle that will be enforced by using the guidance law derived in [8]. This cooperative guidance law minimizes the following cost function

$$J = \frac{\alpha_1}{2}\xi^2_{d_1 m}(t^f_{d_1 m}) + \frac{\alpha_2}{2}\xi^2_{d_2 m}(t^f_{d_2 m}) + \frac{\beta}{2}\left[\gamma_{d_1 m}(t^f_{d_1 m}) - \gamma_{d_2 m}(t^f_{d_2 m}) - \Delta_c\right]^2 \tag{17}$$

$$+\frac{1}{2}\int_0^{t^f_{d_1 m}} u^2_{d_1} dt + \frac{1}{2}\int_0^{t^f_{d_2 m}} \eta^2 u^2_{d_2} dt, \tag{18}$$

where $\xi_{d_i m}$ is the relative displacement between the missile and the i-th defender normal to the LOS used for linearization, Δ_c is the required relative intercept angle, and α_i, η, and β are nonnegative weighs.

For ideal defenders' dynamics (i.e., zero lag), the optimal closed-form solution that minimizes the cost function J was found to have the following form [8]

$$u^*_{d_1}(t) = \frac{N^{u_1}_{Z_1}}{(t^{go}_{d_1 m})^2} Z_1(t) + \frac{N^{u_1}_{Z_2}}{(t^{go}_{d_1 m})^2} Z_2(t) + N^{u1}_{\Delta Z_3}\frac{V_{d_1}}{t^{go}_{d_1 m}}\left(Z_3(t) - Z_4(t) - \Delta_c\right), \qquad t \in [0, t^f_{d_1 m}], \tag{19a}$$

$$u^*_{d_2}(t) = \frac{N^{u_2}_{Z_1}}{(t^{go}_{d_2 m})^2} Z_1(t) + \frac{N^{u_2}_{Z_2}}{(t^{go}_{d_2 m})^2} Z_2(t) + N^{u2}_{\Delta Z_3}\frac{V_{d_2}}{t^{go}_{d_2 m}}\left(Z_3(t) - Z_4(t) - \Delta_c\right), \qquad t \in [0, t^f_{d_2 m}], \tag{19b}$$

where the navigation gains are given in [8].

The ZEM distances of the first and the second defender, Z_1 and Z_2, and the zero-effort flight-path angles (ZEA-s) of the missile plus that of the first and second defender, Z_3 and Z_4, are give by

$$Z_1(t) = \xi_{d_1 m} + \dot{\xi}_{d_1 m} t^{go}_{d_1 m} + k_{d_1 m} a_m (t^{go}_{d_1 m})^2/2, \tag{20a}$$

$$Z_2(t) = \xi_{d_2 m} + \dot{\xi}_{d_2 m} t^{go}_{d_2 m} + k_{d_2 m} a_m (t^{go}_{d_2 m})^2/2, \tag{20b}$$

$$Z_3(t) = \gamma_{d_1} + \gamma_m + t^{go}_{d_1 m} a_m / V_m, \tag{20c}$$

$$Z_4(t) = \gamma_{d_2} + \gamma_m + t^{go}_{d_2 m} a_m / V_m. \tag{20d}$$

where the linearization parameter k_{im} satisfies

$$k_{im} = \cos(\gamma_{m;0} + \lambda_{im;0}), \qquad i \in \{d_1, d_2\}, \tag{21}$$

and relates the missile acceleration a_m to the direction normal to LOS used for the linearization.

Note that the ZEM distances and the ZEA-s of Eq. (20) are valid only under the assumption that the missile is maintaining a known constant maneuver throughout the engagement, i.e., $u_m = const$. Additionally, the guidance law of Eq. (19) was derived under the assumption that the future missile maneuver as well as the relative states are known or are accurately measured, see [8] for further details.

Remark 1. The cost function J enforces an explicit cooperation between the defenders, as their trajectories are mutually dependent on each other. Letting $\alpha_i \to \infty$ yields a perfect intercept between the i-th defender and the missile. Similarly, letting $\beta \to \infty$ enforces a perfect relative intercept angle Δ_c. The parameter η controls the ratio between the weight on the first and the second defender's control effort.

Target's Implicit Cooperation and Prediction of the Missile's Acceleration Profile

As the missile is homing onto the target that performs an evasive maneuver, it is apparent that the constant missile acceleration assumption is invalidated in the scenario studied in this paper. Moreover, we consider a highly uncertain environment where the only interference with the external environment is thanks to the two noisy LOS measurements. All these emerged problems are to be discussed in the sequel.

The implicit cooperation of the defenders' guidance law stems from the fact that the defenders are aware of the evasive maneuver of the target and thus can anticipate the maneuvers it will induce on the incoming homing missile. Thus, for a given missile guidance strategy, this information can be used to obtain a_m as a function of time via numerical integration of the appropriate engagement equations. By doing so, the constant missile acceleration assumption in the defenders' guidance law can be relaxed. Additionally, the potential intercept points of the i-th defender-missile engagement can be predicted, which in turn can help to reduce the defenders' acceleration demand, hence reduce the likelihood of control saturation.

When the future missile maneuver is known but is not constant, i.e., $u_m \neq const.$, then based on the terminal projection transformation, the zero-effort variables of Eq. (20) are generally given by [7, 16]

$$Z(t) = D\Phi^{(i)}(t_{im}^{go})x^{(i)}(t) + D\int_{t}^{t_{im}^{f}} \Phi^{(i)}(t_{im}^{f}, \tau)C^{(i)}u_m d\tau, \qquad i \in \{d_1, d_2\}, \tag{22}$$

where D is a constant row vector that pulls out the appropriate element of the zero-effort variable, $\Phi^{(i)}(t_{im}^{f}, \tau)$ is the transition matrix associated with the homogenous solution of the linearized i-th defender-missile engagement, and $C^{(i)}$ is a vector associated with the linearized one-sided problem. Based on Eq. (22), the computation of the ZEM distances and ZEA-s of Eq. (20) are replaced by the following equations (assuming constant missile speed V_m)

$$Z_1(t) = \xi_{d_1m} + \dot{\xi}_{d_1m}t_{d_1m}^{go} + \int_{t}^{t_{d_1m}^{f}} (t_{d_1m}^{f} - \tau)a_m(\tau)\cos\left(\gamma_m(\tau) + \lambda_{d_1m}(\tau)\right)d\tau, \tag{23a}$$

$$Z_2(t) = \xi_{d_2m} + \dot{\xi}_{d_2m}t_{d_2m}^{go} + \int_{t}^{t_{d_2m}^{f}} (t_{d_2m}^{f} - \tau)a_m(\tau)\cos\left(\gamma_m(\tau) + \lambda_{d_2m}(\tau)\right)d\tau, \tag{23b}$$

$$Z_3(t) = \gamma_{d_1} + \gamma_m + \int_{t}^{t_{d_1m}^{f}} \dot{\gamma}_m(\tau)d\tau = \gamma_{d_1} + \gamma_m + \frac{1}{V_m}\int_{t}^{t_{d_1m}^{f}} a_m(\tau)d\tau, \tag{23c}$$

$$Z_4(t) = \gamma_{d_2} + \gamma_m + \int_{t}^{t_{d_2m}^{f}} \dot{\gamma}_m(\tau)d\tau = \gamma_{d_2} + \gamma_m + \frac{1}{V_m}\int_{t}^{t_{d_2m}^{f}} a_m(\tau)d\tau, \tag{23d}$$

where the integral components in Eq. (23) are computed by numerical integration and time propagation of the relevant parts of Eqs. (2) and (4), assuming that no further acceleration commands are issued by the defenders and that the missile and the target follow the presumed maneuvering model. If a_m is constant throughout the engagement, then Eq. (23) degenerates to Eq. (20).

Implementation Issues

The variables $\xi_{d_i m}$ and $\dot{\xi}_{d_i m}$, which appear in Z_1 and Z_2 of Eq. (23), relate to the linearized model. To implement the defenders' guidance law in a nonlinear setting, we need to replace these variables by more meaningful kinematic variables. Assuming small deviations from collision triangle, the displacement $\xi_{d_i m}$ can be reasonably well approximated by

$$\xi_{d_i m} \approx \rho_{d_i m}(\lambda_{d_i m} - \lambda_{d_i m;0}), \tag{24}$$

where $\lambda_{d_i m;0}$ is the LOS angle used for linearization. Differentiating Eq. (24) with respect to time yields

$$\xi_{d_i m} + \dot{\xi}_{d_i m} t^{go}_{d_i m} = -V_{\rho d_i m} \dot{\lambda}_{d_i m} (t^{go}_{d_i m})^2. \tag{25}$$

The left hand side of Eq. (25) is identical to the first two terms of Eqs. (23a) and (23b), respectively. By this, $\xi_{d_i m}$ and $\dot{\xi}_{d_i m}$ are replaced by $\dot{\lambda}_{d_i m}$ and $V_{\rho d_i m}$, defined in Eqs. (2) and (3a), respectively.

Due to the same assumption, the speed $V_{\rho d_i m}$ can be assumed constant, and the $t^{go}_{d_i m}$, defined in (7), can be approximated by

$$t^{go}_{im} \approx \begin{cases} -\rho_{im}/V_{\rho im}, & V_{\rho im} < 0 \\ 0, & V_{\rho im} \geq 0 \end{cases}, \qquad i \in \{d_1, d_2\}. \tag{26}$$

IV. Joint Estimator Design

To properly implement the defenders' cooperative guidance law presented in Sec. III.C,we need the zero-effort miss/angle variables, time-to-go, and the relative geometry associated with the defenders. In real-world scenarios, these variables cannot be measured and therefore need to be estimated. Note that such variables may be estimated independently by each defender or cooperatively, as done for attacking missiles in [1–3]. In these works, it was assumed that each missile has its own estimator and the computed state estimates are shared within the team. Such approach requires extra computational effort because the parameters directly related to the opponent are redundantly estimated by each entity in the team. In this work, these redundantly estimated parameters correspond to the missile's acceleration, flight path angle, speed, and pertinent guidance parameters, respectively.

IV.A. Estimation Model and Assumptions

We assume that the missile has no information about the defenders, it is not trying to evade the defenders, and it is guided towards the target via one of the classical guidance laws of PN, APN, or OGL with fixed guidance parameter N'_{PN}, N'_{APN}, or α, see Sec. III.A for more details. If the missile's active guidance strategy is fixed throughout the engagement, a static multiple-model approach can be used to identify the guidance law and guidance parameters [13, 14]. If the missile switches between guidance strategies, a dynamic multiple-model approach can be derived based on the interactive multiple-model (IMM) approach [1, 3].

In this paper, we assume that, prior to launching the defending missiles, the target successfully identified the active guidance law of the missile and passed this information to the defenders. However, we assume that the corresponding parameters of the missile guidance law are still unknown. These uncertain guidance parameters must be estimated together with other (uncertain) variables, which are all used to properly implement the defenders' guidance strategy.

The i-th defender's state vector of the missile in polar coordinates is

$$x^R_{d_i m} = \begin{bmatrix} \rho_{d_i m} & \lambda_{d_i m} & \gamma_m & x^a_m & V_m & \delta_m \end{bmatrix}^T, \tag{27}$$

where δ_m represents the unknown guidance parameter(s). In our case, δ_m may stand for N' or α, depending on the considered guidance law of the missile.

In this paper, instead of designing two estimators, one for $x^R_{d_1 m}$ and the other for $x^R_{d_2 m}$, we design a single estimator for the joint defenders' state, defined as

$$x^R_{dm} = \begin{bmatrix} \rho_{d_1 m} & \rho_{d_2 m} & \lambda_{d_1 m} & \lambda_{d_2 m} & x^a_m & \gamma_m & V_m & \delta_m \end{bmatrix}^T. \tag{28}$$

It is obvious that $\dim(x_{dm}^R) < \dim(x_{d_1m}^R) + \dim(x_{d_2m}^R)$. In the rest of the paper, to avoid excessive indexing, we will represent x_{dm}^R as x.

Assume that the parameters of the missile dynamics are known, then the model used for estimation is given by

$$\begin{cases} \dot{\rho}_{d_1m} = V_{\rho d_1 m} \\ \dot{\rho}_{d_2m} = V_{\rho d_2 m} \\ \dot{\lambda}_{d_1m} = V_{\lambda d_1 m}/\rho_{d_1m} \\ \dot{\lambda}_{d_2m} = V_{\lambda d_2 m}/\rho_{d_2m} \\ \dot{x}_m^a = A_m x_m^a + B_m u_m(x_{tm}^R, \delta_m) \\ \dot{\gamma}_m = \left(C_m x_m^a + D_m u_m(x_{tm}^R, \delta_m)\right)/V_m \\ \dot{V}_m = 0 \\ \dot{\delta}_m = 0 \end{cases}, \tag{29}$$

where $V_{\rho d_i m}$ and $V_{\lambda d_i m}$ are given in Eqs. (3a) and (3b), respectively, and $u_m(x_{tm}^R, \delta_m)$ is the missile's acceleration command given by (10). This command depends on the relative state between the target and the missile

$$x_{tm}^R = \begin{bmatrix} \rho_{tm} & \lambda_{tm} & a_t & \gamma_t \end{bmatrix}^T \tag{30}$$

and the active guidance strategy of the missile.

For simplicity, we assume that the missile has perfect information about the target, but not vice versa. Therefore, to use x_{tm}^R in Eq. (29), we need to compute x_{tm}^R using information that is available to the defender team. As $x_i^I,\ i \in \{t, d_1, d_2\}$ are assumed to be known, we can therefore express ρ_{tm} and λ_{tm} as a function of the most recent estimates of $\rho_{d_i m}$ and $\lambda_{d_i m}$, i.e.,

$$\rho_{tm} = \frac{\sqrt{(\Delta X_{tm}^{d_1})^2 + (\Delta Y_{tm}^{d_1})^2} + \sqrt{(\Delta X_{tm}^{d_2})^2 + (\Delta Y_{tm}^{d_2})^2}}{2}, \tag{31a}$$

$$\lambda_{tm} = \frac{\operatorname{atan2}\left(\Delta Y_{tm}^{d_1}, \Delta X_{tm}^{d_1}\right) + \operatorname{atan2}\left(\Delta Y_{tm}^{d_2}, \Delta X_{tm}^{d_2}\right)}{2}, \tag{31b}$$

where $\Delta X_{tm}^{d_i}$ is the horizontal and $\Delta Y_{tm}^{d_i}$ is the vertical separation between the target and the missile from the i-th defender's perspective, respectively, given by

$$\Delta X_{tm}^{d_i} = \Delta x_{td_i} + \rho_{d_i m}\cos(\lambda_{d_i m}), \qquad \Delta x_{td_i} \triangleq x_{d_i} - x_t, \tag{32a}$$

$$\Delta Y_{tm}^{d_i} = \Delta y_{td_i} + \rho_{d_i m}\sin(\lambda_{d_i m}), \qquad \Delta y_{td_i} \triangleq y_{d_i} - y_t. \tag{32b}$$

It is evident that both ρ_{dm} and λ_{dm} in Eq. (31) are computed as arithmetic averages of the two perspectives. By this approach, the robustness of the proposed estimation scheme is increased, because the effect of deteriorating estimation accuracy from one "perspective" can be averaged out by the possible accurate estimate from the other perspective. As a_t and γ_t are assumed to be known, x_{tm}^R is now fully defined.

Let us denote the vector that contains all the target-defenders relative positions at time t_k as

$$x_{td;k}^R = [\Delta x_{td_1}\ \Delta y_{td_1}\ \Delta x_{td_2}\ \Delta y_{td_2}]^T. \tag{33}$$

By this, the discrete-time version of Eq. (29), used for the estimator design, can be compactly rewritten as

$$x_k = f_{k-1}(x_{k-1}, x_{td;k}^R), \tag{34}$$

where x_k is the defenders' joint state vector x_{dm}^R at time t_k, and f_{k-1} is a vector function derived by integrating of Eq. (29) from t_{k-1} to t_k.

IV.B. Combined Measurement Model and Information Sharing

The defenders form a measuring baseline relative to the missile in space. Different look angles of the defenders on the missile can improve the observability of the information-sharing based estimation scheme.

By exploiting the triangulation technique from the measurements perspective, we can express the model of the i-th physical measurement, given in Eq. (9), as a function of the other, j-th defender, variables and the known relative position of the two defenders

$$z_{i;k} = h_j^{\dagger}(x_k, x_{dd;k}^R) + v_i = \text{atan2}(\Delta Y_{ji}, \Delta X_{ji}) + v_i, \quad i, j \in \{d_1, d_2\} \wedge i \neq j \tag{35}$$

where $x_{dd;k}^R = [\Delta x_{d_1 d_2} \ \Delta y_{d_1 d_2} \ \Delta x_{d_2 d_1} \ \Delta y_{d_2 d_1}]^T$, and

$$\Delta X_{ji} = \Delta x_{ji} + \rho_{jm} \cos(\lambda_{jm}), \qquad \Delta x_{ji} \triangleq x_i - x_j, \tag{36a}$$

$$\Delta Y_{ji} = \Delta y_{ji} + \rho_{jm} \sin(\lambda_{jm}), \qquad \Delta y_{ji} \triangleq y_i - y_j. \tag{36b}$$

Combining the physical measurement model of Eq. (9) with the indirect measurement model of Eq. (35) yields to the combined measurement model

$$z_k = \begin{bmatrix} z_{d_1;k} \\ z_{d_2;k} \\ z_{d_1;k} \\ z_{d_2;k} \end{bmatrix} = h(x_k, x_{dd;k}^R) + v_k = \begin{bmatrix} h_1(x_k) \\ h_2(x_k) \\ h_2^{\dagger}(x_k, x_{dd;k}^R) \\ h_1^{\dagger}(x_k, x_{dd;k}^R) \end{bmatrix} + v_k \tag{37}$$

where $v_k = [v_{d_1;k} \ \ v_{d_2;k} \ \ v_{d_1;k} \ \ v_{d_2;k}]^T$, $z_{d_i;k}$ is the physical LOS angle measurement of the i-th defender, and the functions h_i and $h_i^{\dagger}$ are defined in Eqs. (9) and (35), respectively.

When the i-th defender passes the missile, i.e., $t_{d_i m}^{go} = 0$, this defender does not transmit any measurements to the other defender and hence only a single physical model is considered for measurement update. In the next section, we will use the combined measurement model of Eq. (37) to design the joint estimator.

IV.C. Extended Kalman Filter

As the estimation model in Eq. (29) is nonlinear, an extended Kalman filter (EKF) will be used to estimate the state vector defined in Eq. (28). Note, however, that other estimation methods such as various variants of the Kalman filters, divided difference filters, particle filters, to name just a few, can be also appropriate.

The state estimate of the filter at time t_k using measurements up to time t_{k-1}, $\hat{x}_{x|k-1}$, is propagated in time using Eq. (34) and the most up-to-dated $x_{td;k}^R$. The state transition matrix $\Phi_{k|k-1}$ associated with the system dynamics of Eq. (29) can be approximated by

$$\Phi_{k|k-1} = \exp(F_{k-1|k-1}T) \approx I + F_{k-1|k-1}T, \tag{38}$$

where $T = t_k - t_{k-1}$ is the sampling time used for time propagation, I is the identity matrix of appropriate dimension, and $F_{k-1|k-1}$ is the Jacobian matrix associated with the dynamics of Eq. (34), i.e.,

$$F_{k-1|k-1} = \left. \frac{\partial f_{k-1}(x, x_{td}^R)}{\partial x} \right|_{x=\hat{x}_{k-1|k-1}}, \tag{39}$$

is assumed to be fixed during the time interval $(t_{k-1}, t_k]$. The prediction error covariance matrix is

$$P_{k|k-1} = \Phi_{k|k-1} P_{k-1|k-1} \Phi_{k|k-1}^T + Q_d, \tag{40}$$

where Q_d is an artificial covariance matrix of the corresponding discrete process noise used as a tuning parameter of the filter [17].

If the physical measurement from the i-th defender, $z_{i;k}$, is not available (e.g., because the i-th defender ceased to exist, or due to sensor error, blind range of the sensor, etc.), then the time propagated state estimate $\hat{x}_{k|k-1}$ is updated by

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \left(z_k - h(\hat{x}_{k|k-1}) \right), \tag{41}$$

where K_k is the Kalman gain computed as

$$K_k = P_{k|k-1} H_k^T \left(H_k P_{k|k-1} H_k^T + R \right)^{-1}. \tag{42}$$

with $j \neq i$

$$H_k = \begin{bmatrix} 0 & \dots & 1^{(2+j)} & \dots & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} \sigma^2_{\lambda d_j} \end{bmatrix} \tag{43}$$

where the index $(2+j)$ indicates the location of the only nonzero element of the vector H_k.

If the estimation is performed using measurements from both defenders, the measurement Jacobian matrix H_k and the measurement noise covariance matrix R becomes

$$H_k = \left. \begin{bmatrix} 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ 0 & H^\rho_{d_2 d_1} & 0 & H^\lambda_{d_2 d_1} & [0]_{4\times 4} \\ H^\rho_{d_1 d_2} & 0 & H^\lambda_{d_1 d_2} & 0 & \end{bmatrix} \right|_{x=\hat{x}_{k|k-1}}, \qquad R = \text{diag}\left\{ \sigma^2_{\lambda d_1}, \sigma^2_{\lambda d_2}, \sigma^2_{\lambda d_1}, \sigma^2_{\lambda d_2} \right\}, \tag{44}$$

where

$$H^\rho_{ji} = \frac{\Delta x_{ji} \sin(\lambda_{jm}) - \Delta y_{ji} \cos(\lambda_{jm})}{\Lambda_{ji}}, \qquad H^\lambda_{ji} = \frac{\left(\Delta x_{ji} \cos(\lambda_{jm}) + \Delta y_{ji} \sin(\lambda_{jm}) + \rho_{jm}\right) \rho_{jm}}{\Lambda_{ji}}, \tag{45}$$

and the common denominator is

$$\Lambda_{ji} = \Delta x^2_{ji} + \Delta y^2_{ji} + \rho^2_{jm} + 2\rho_{jm} \left[\Delta x_{ji} \cos(\lambda_{jm}) + \Delta y_{ji} \sin(\lambda_{jm}) \right].$$

Finally, the covariance matrix is updated using

$$P_{k|k} = P_{k|k-1} - K_k H_k P_{k|k-1}, \tag{46}$$

V. Numerical Analysis

In this section, we analyze the ability of the cooperative target-defenders team to protect the targeted aircraft from the attacking missile. We also study the effect of different values of the commanded relative intercept angle, Δ_c, on the estimation as well as on the intertwined guidance-estimation problem.

V.A. Engagement Scenario and Simulation Environment

We consider a similar engagement scenario as presented in [14]. All engagements are initiated at a horizontal separation of 5 km between the target and the missile. To model the separation effect, the defenders are initiated at a vertical separation of $\Delta y_{td_1} = \Delta y_{td_2} = -1$ m below the target. The defenders are launched from the aircraft at the beginning of the engagement ($t = 0$ s). The target's speed is $V_t = 300$ m/s and the speed of the two defenders and the missile is equal and is $V_{d_1} = V_{d_2} = V_m = 500$ m/s. For the analysis, it is assumed that the missile and the target have first-order strictly proper dynamics with time constants $\tau_m = 0.2$ s and $\tau_t = 0.5$ s. Thus, matrices in Eq. (4) degenerate to $A_i = -1/\tau_i$, $B_i = 1/\tau_i$, $C_i = 1$, and $d_i = 0$, $i \in \{m, t\}$. We consider ideal dynamics for both defenders. The target's maneuver capability is limited to $a_t^{max} = 5$ g. No saturation is applied on the missile's acceleration command, i.e., $a_m^{max} = \infty$ g. The defenders' maneuverability belongs to the closed set $a_d^{max} \in \mathcal{U}_d^{max} = \{10, 20, 30, 40, \infty\}$ g, where g is the standard acceleration due to the gravity (g = 9.80665 m/s^2). Note, the omitted subscript "i" in a_d^{max} indicates that both defenders are equally concerned, i.e., $a_d^{max} = a_{d_1}^{max} = a_{d_2}^{max}$. We will use the same notation simplification for other variables in the next. The missile's initial flight path angle is chosen such that the missile's velocity vector points towards the initial target location, i.e., $\gamma_{m;0} = 0$ deg. As the defenders are launched from the aircraft's platform, therefore the initial flight path angles of the defenders are considered to be identical to the initial flight path angle of the target, i.e., $\gamma_{d;0} = \gamma_{t;0}$. For the closed loop MC analysis, these angles are drawn from uniform distribution of the closed interval $[-30, 30]$ deg.

The missile is guided towards the target using PN guidance with $N' = 4$ and using perfect information. We assumed that throughout the engagement the target applies maximum acceleration to one side. The maneuver direction is determined based on the initial geometry as

$$u_t = \begin{cases} +a_t^{max} & \text{if} \quad \gamma_{t;0} \geq 0, \\ -a_t^{max} & \text{if} \quad \gamma_{t;0} < 0, \end{cases}$$

where $\gamma_{t;0}$ is the initial flight path angle of the target defined above. The defenders use the cooperative guidance law of Eq. (19) with implicit target cooperation, see Eq. (23). The numerical values of the guidance parameters are $\alpha_1 = \alpha_2 = 10^5$, $\beta = 10^8$, and $\eta = 1$. The states needed for the defenders' guidance law employment are estimated at each time step, using the estimator developed in Sec. IV, at a sampling rate of 50 Hz ($T = 1/50$ s). The simulated measurement noises are with $\sigma_{\lambda_{dm}} = 1$ mrad. The filter's tuning parameter Q_d has been chosen by numerical simulations. The initial state of the filter is sampled from a Gaussian distribution

$$\hat{x}_{0|0} \sim \mathcal{N}(x_0, P_{0|0}),$$

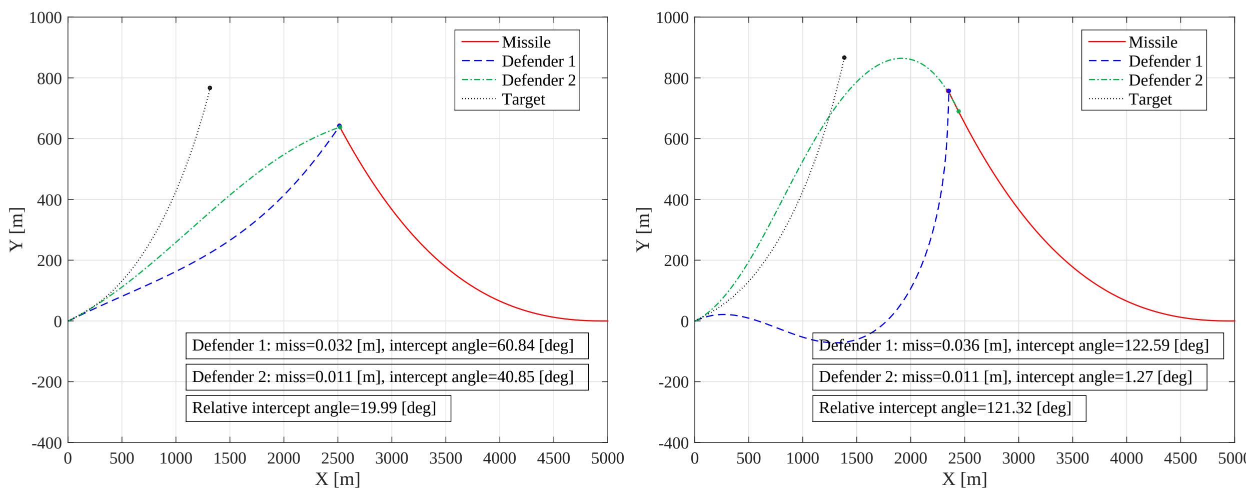
where x_0 is the true state vector and $P_{0|0}$ is the initial covariance matrix of the error given by

$$P_{0|0} = \text{diag}\left\{100^2 \quad 100^2 \quad (5\pi/180)^2 \quad (5\pi/180)^2 \quad (2.5\text{g})^2 \quad (5\pi/180)^2 \quad 50^2 \quad 2^2\right\}. \tag{47}$$

V.B. Sample Run Example

Two different commanded relative intercept angles $\Delta_c = 20$ deg and $\Delta_c = 120$ deg are considered for sample run demonstration. The initial flight path angle of the target-defender team is $\gamma_{i;0} = 10$ deg. The defenders are guided towards the missile using perfect information (true state vector) and no maneuverability limitation. The same assumptions hold for the missile.

Figure 2 and 3 present the planar trajectories and the acceleration profiles of the target, missile, and the two defenders in the simulated sample runs, respectively. It can be seen that, although the there is a requirement on a specific intercept angle of $\Delta_c = 20$ deg, the maximal acceleration requirement from the defenders is quite small, approx. 6 g, compared to the missile's maximal acceleration being above 7 g. On the other hand, as seen in Fig. 3b, significantly larger relative intercept angle requirement naturally leads to much higher maneuverability requirements from the defenders.



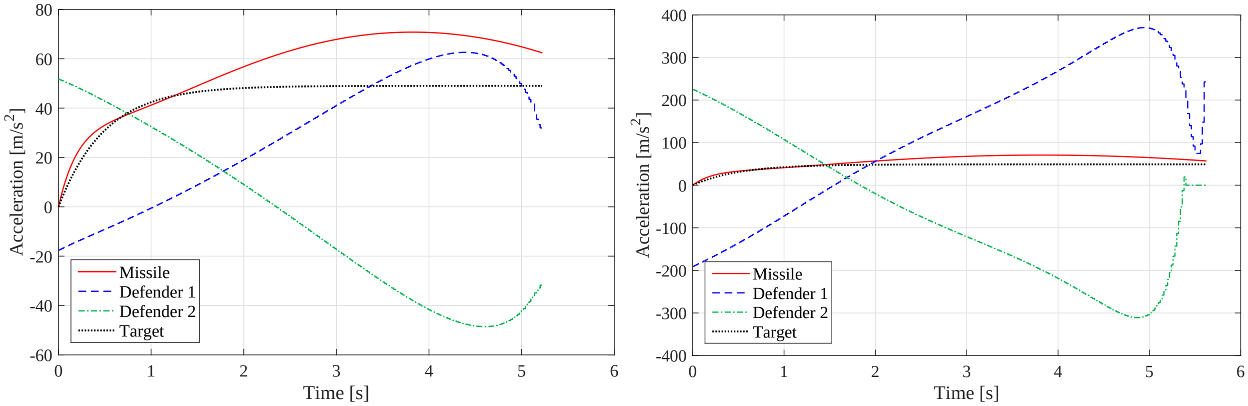
(a) $\Delta_c = 20$ [deg] (b) $\Delta_c = 120$ [deg]

Figure 2: Sample trajectories for different relative intercept angles.

V.C. Guidance Performance Evaluation in Closed Loop

The effect of different values of $\Delta_c \in \mathcal{D}_c$ on the intertwined guidance-estimation problem is analyzed here. The analysis is done for various considerations of the defenders maneuverability limit $a_d^{max} \in \mathcal{U}_d^{max}$. The estimated state from the estimator is used to guide the defenders towards the missile. For comparison, full information and saturation-free case is also examined. This case is referred in the next figures as "$u_d^{max} = \infty$ g & perfect state". For each value of Δ_c and u_d^{max}, a set of 500 MC simulations was run. The guidance performance is evaluated in terms of the achieved miss distance and acceleration requirement.

In case of the miss distance, we first compute the "two defender" cumulative distribution function (CDF), which is defined on the minimum miss of both defenders. Then, using this CDF, we compute the value of



(a) $\Delta_c = 20$ [deg]

(b) $\Delta_c = 120$ [deg]

Figure 3: Sample acceleration profiles for different relative intercept angles.

the miss which corresponds to the 95% of cases, i.e., Prob(miss) ≤ 0.95. This quantity is denoted as $\text{miss}_{95\%}$ and is also known as warhead lethality range which ensures a 95% kill probability for the defender team. To evaluate the maneuverability requirements, we consider the value of the two defender maximal acceleration in 95% of the simulation campaign cases. We denote this value as $a_d^{max}(95\%)$. This value is computed analogously as $\text{miss}_{95\%}$ is computed. Additionally to $a_d^{max}(95\%)$, we also consider a running cost J_{acc} on the acceleration profiles defined as $J_{acc} = \int_0^{t^f_{d_1m}} |a_{d_1}(\tau)|d\tau + \int_0^{t^f_{d_2m}} |a_{d_2}(\tau)|d\tau$.

Figure 4 presents the obtained results of $\text{miss}_{95\%}$ for different intercept angles $\Delta_c \in \mathcal{D}_c$ and acceleration limits $u_d^{max} \in \mathcal{U}_d^{max}$. Figure 5a shows the maneuverability requirements of the defenders in terms of the $a_d^{max}(95\%)$ measure and Fig. 5b in terms of the running cost J_{acc} measure. The results of Figs. 4-5 suggest that small values of Δ_c yield to larger miss distances as the estimation performance for these angles is poor. Due to the same reason, such small angles also cause increase of "momentary" maximal acceleration requirements. On the other hand, large values of Δ_c require substantially more "overall" maneuverability yielding to control saturation. Obviously, long-term saturation has large effects on the achievable miss, even when accurate estimates are used. It is interesting to note that for all finite maneuverability limit cases, i.e., for all $u_d^{max} < \infty$, there exist a plateau effect, i.e., a region of intercept angles where the obtained miss is minimal. As expected, the performance of any perfect information guidance law is better than the performance of the same guidance law using estimated states, see the results for $u_d^{max} = \{\infty, \infty^\star\}$ in Fig. 4.

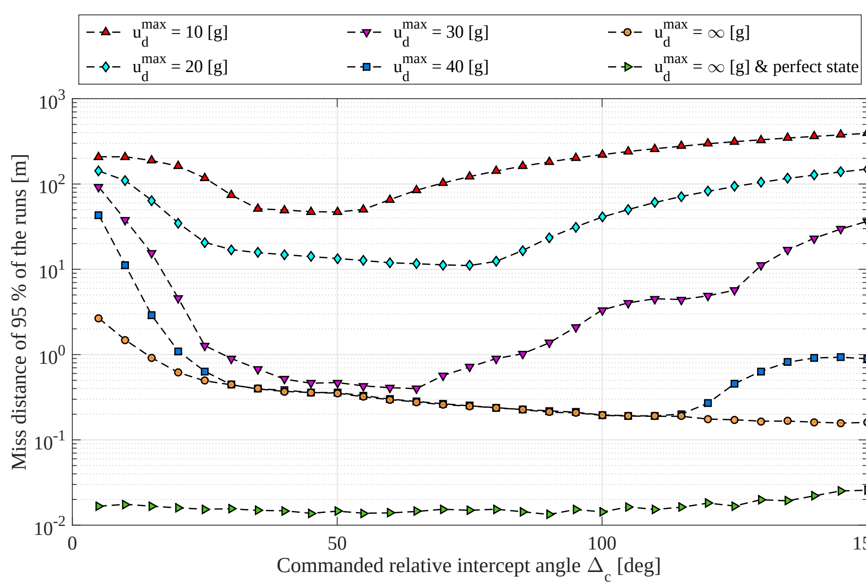
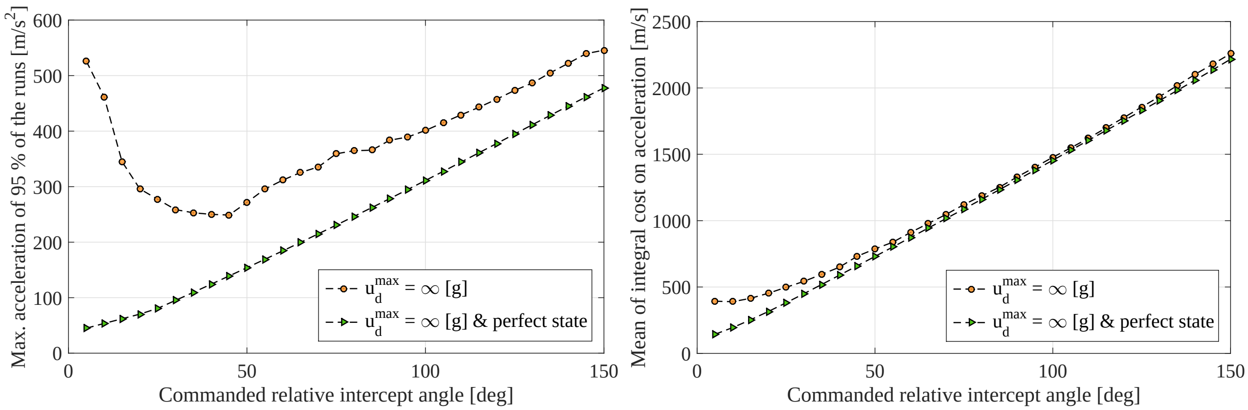


Figure 4: Values of $\text{miss}_{95\%}$ for $u_d^{max} \in \mathcal{U}_d^{max}$ as a function of $\Delta_c \in \mathcal{D}_c$.



(a) Values of $a_d^{max}(95\%)$ as a function of $\Delta_c \in \mathcal{D}_c$.

(b) Values of J_{acc} as a function of $\Delta_c \in \mathcal{D}_c$.

Figure 5: Guidance performance - acceleration requirements for unsaturated case.

VI. Conclusions

A cooperative estimation-guidance algorithm has been presented for a team of two aircraft's defending missiles to intercept an attacking missile homing on to the evading target aircraft. The algorithm exploits an explicit team cooperation in the defenders' guidance to impose a relative intercept angle, an implicit cooperation of the target, and a cooperative estimation scheme based on shared information. The cooperation from the target's point of view results from the fact that the defenders are aware of the evasion maneuvers of the target. Thus, the defenders can predict the target-induced maneuvers on the homing missile.

The proposed joint estimation scheme is strongly linked to the defenders' guidance when considering LOS angle measurements only. Nonlinear simulations revealed that various relative intercept angle constraints for the defenders have strong influence on the intertwined guidance-estimation performance of the defenders. Small angles yield to poor estimation performance which consequently lead to control saturation and intercept performance degradation. Angles ranging from approx. 20 deg to approx. 60 deg exhibit very good guidance performance while maintaining modest maneuverability requirements. Larger intercept angles lead only to negligible improvements in the interception performance but require far more agility from the defenders.

The effectiveness demonstrated by the proposed cooperative algorithm to protect the evading aircraft from a highly maneuverable homing missile can, for a carefully selected relative intercept angle, considerably improve the aircraft's survivability, making it possible to design relatively inexpensive defending missiles, without having superior maneuverability requirements, advanced sensor systems, and large lethal warhead.

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References

[1] Shaferman, V. and Oshman, Y., "Cooperative interception in a multi-missile engagement," *AIAA Guidance, Navigation, and Control Conference and Exhibit*, Chicago, IL, Aug. 2009, AIAA Paper 2009-5783.

[2] Liu, Y., Qi, N., and Shan, J., "Cooperative Interception with Double-Line-of-Sight-Measuring," *AIAA Guidance, Navigation, and Control Conference*, AIAA, Boston, MA, 2013, AIAA Paper 2013-5112.

[3] Shaferman, V. and Oshman, Y., "Stochastic Cooperative Interception Using Information Sharing Based on Engagement Staggering," *Journal of Cuidance, Control, and Dynamics*, 2016.

[4] Chen, T. and Xu, S., "Double line-of-sight measuring relative navigation for spacecraft autonomous

rendezvous," *Acta Astronautica*, Vol. 67, No. 1, 2010, pp. 122–134.

[5] Chen, T. and Xu, S., "Approach guidance with double-line-of-sight measuring navigation constraint for autonomous rendezvous," *Journal of Guidance, Control, and Dynamics*, Vol. 34, No. 3, 2011, pp. 678–687.

[6] Ryoo, C.-K., Cho, H., and Tahk, M.-J., "Optimal guidance laws with terminal impact angle constraint," *Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 4, 2005, pp. 724–732.

[7] Shaferman, V. and Shima, T., "Linear quadratic guidance laws for imposing a terminal intercept angle," *Journal of Guidance, Control, and Dynamics*, Vol. 31, No. 5, 2008, pp. 1400–1412.

[8] Shaferman, V. and Shima, T., "Cooperative optimal guidance laws for imposing a relative intercept angle," *Journal of Guidance, Control, and Dynamics*, Vol. 38, No. 8, 2015, pp. 1395–1408.

[9] Yuan, L. C., "Homing and Navigational Courses of Automatic Target Seeking Devices," *Journal of Applied Physics*, Vol. 19, No. 12, 1948, pp. 1122–1128.

[10] Garber, V., "Optimum Intercept Laws for Accelerating Targets," *AIAA Journal*, Vol. 6, No. 11, 1968, pp. 2196–2198.

[11] Cottrell, R. G., "Optimal Intercept Guidance for Short-Range Tactical Missiles," *AIAA journal*, Vol. 9, No. 7, 1971, pp. 1414–1415.

[12] Shima, T., "Optimal cooperative pursuit and evasion strategies against a homing missile," *Journal of Guidance, Control, and Dynamics*, Vol. 34, No. 2, 2011, pp. 414–425.

[13] Fonod, R. and Shima, T., "Multiple Model Adaptive Evasion Against a Homing Missile," *Journal of Guidance, Control, and Dynamics*, Vol. 39, No. 7, 2016, pp. 1578–1592.

[14] Shaferman, V. and Shima, T., "Cooperative multiple-model adaptive guidance for an aircraft defending missile," *Journal of Guidance, Control, and Dynamics*, Vol. 33, No. 6, 2010, pp. 1801–1813.

[15] Prokopov, O. and Shima, T., "Linear quadratic optimal cooperative strategies for active aircraft protection," *Journal of Guidance, Control, and Dynamics*, Vol. 36, No. 3, 2013, pp. 753–764.

[16] Gutman, S., *Applied min-max approach to missile guidance and control*, Vol. 209 of *Progress in Astronautics and Aeronautics*, AIAA, Reston, VA, 2005, Chap. 9.

[17] Bar-Shalom, Y., Li, X. R., and Kirubarajan, T., *Estimation with applications to tracking and navigation: theory algorithms and software*, John Wiley & Sons, Inc., New York, NY, 2001, Chap. 10.